MATH 147 QUIZ 11 SOLUTIONS

1. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, for $\mathbf{F} = -\frac{1}{2}x\vec{i} + -\frac{1}{2}y\vec{j} + \frac{1}{4}\vec{k}$ for C the helix given by $\mathbf{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + t\vec{k}$, from the point $(1,0,0)$ to $(-1,0,3\pi)$. (5 Points)

   We proceed by calculating $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b F(x(t),y(t),z(t))\cdot\mathbf{r}'(t)dt$. Thus, we find that $\mathbf{r}'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j} + \vec{k}$, and $F(t) = -1/2\cos(t)\vec{i} - 1/2\sin(t)\vec{j} + 1/4\vec{k}$, and so the dot product of F and r' is $\frac{1}{2}\sin(t)\cos(t) - \frac{1}{2}\sin(t)\cos(t) + \frac{1}{4} = \frac{1}{4}$. Next, we note that we can find the bounds of t by looking at the z coordinate, which informs us that $a = 0$ and $b = 3\pi$. Thus, our integral is

$$\int_0^{3\pi} \frac{1}{4}dt = \frac{3\pi}{4}.$$

2. Verify Gauss's Theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_B \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV,$$

   for $\mathbf{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and S the sphere of radius R centered at the origin. Use the outward pointing normal vector.(5 points)

   We compute the surface integral without parameterizing S. A sphere of radius R is the level surface $x^2 + y^2 + z^2 = R^2$, and so a normal vector is $2x\vec{i} + 2y\vec{j} + 2z\vec{k}$, which we make unit, giving us unit normal vector $\frac{1}{R}(x\vec{i} + y\vec{j} + z\vec{k})$. Now, we know that the surface integral is defined by

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint (\mathbf{F}\cdot n)\, dS$$

   and we will compute it this way. Note that $\mathbf{F}\cdot n = \frac{1}{R}(x^2+y^2+z^2) = R$, and so $\iint(\mathbf{F}\cdot n)\, dS = R\iint_s dS = R\cdot \text{Area}(S) = 4\pi R^3$.

   Now, to compute the right hand side, we perform a normal triple integral. The sum of the partials, aka the divergence is $\text{div}(F) = 3$, and so the right hand side is

$$\iiint_B 3dV = 3\iiint_b dV = 3\cdot\text{Vol}(B) = 3\cdot\frac{4\pi R^3}{3} = 4\pi R^3.$$

   As the right and left hand sides are the same, we have verified Gauss's Divergence Theorem for the sphere with the given vector field.